MOTION OF A VISCOUS FLUID IN THE INITIAL SECTION OF A FLAT CHANNEL

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Inzhenerno-Fizicheskii Zhurnal, Vol. 12, No. 2, pp. 219–226, 1967

UDC 532.54

The laminar flow of a viscous incompressible fluid in the initial section of a flat channel is investigated on the basis of the linearized boundary layer equations. Formulas are obtained for the longitudinal velocity component and the friction at the wall. The approximate and exact solutions are compared, and the agreement is shown to be satisfactory.

In spite of the fact that with the development of computer technology it has become possible, in principle, to solve problems of laminar flow without resorting to any simplifications, it is still important to have approximate methods of obtaining results rapidly and representing them in analytic form particularly in connection with engineering calculations.

Numerous papers have been devoted to calculation of the flow in the initial sections of flat channels and circular tubes. The problem of liquid flow in the initial section of a tube remote from the inlet was first solved in [1]. The axial velocity was represented in the form of a sum of the Poiseuille solution and a perturbing term. Consequently, the solution could be applied only at a considerable distance from the tube inlet.

In relation to the flow in the initial section, in [2, 3] Schiller suggested that the velocity distribution in the boundary layer is parabolic, and that for flow in a circular tube the initial section ends where the boundary layers join forming a parabolic velocity distribution. Schiller's method was subsequently refined in [4, 5].

The flow in the initial section of a flat channel was examined by Leibenzon in [6] using a method analogous to that of [2, 3]. The same problem was also solved in [7] by combining the solution obtained by means of a frontal approach using boundary layer methods and the solution obtained by approach from behind, which also goes over into the parabolic solution at large x.

A detailed investigation of the flow in the initial sections of tubes, channels, and diffusers on the basis of approximate equations of motion was carried out by S. M. Tagg [8], who solved the problem by employing Oseen's equations

$$U\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial p}{\partial y}=0,\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1}$$

The same equations are used to solve analogous problems in [9, 10].

If the friction stress on a plate is calculated from the solutions of these equations and compared with the exact value, the difference is 65%. Hence it follows that Eqs. (1) are rough approximations, giving exaggerated values of the viscosity stress.

A number of more recent papers have been devoted to numerical methods of solving the problem of the flow of a viscous liquid in the initial section of flat and circular channels.

In [11] a numerical method of solving the boundary layer equations is used as a basis for calculating the flow in the initial section of a flat channel, and the velocity distribution in different sections along its length is obtained. A comparison of the result of this study and the data of [7] indicates satisfactory agreement.

The problem of the flow of a viscous liquid close to the channel inlet is refined in [12]. The fact is that close to the inlet the boundary layer equations are inapplicable, since the derivative $\partial^2u/\partial x^2$ becomes commensurable with $\partial^2u/\partial y^2$, while the pressure gradient across the channel is considerably different from zero, so that it is necessary to obtain a joint solution of the equations of motion for the longitudinal and transverse components of velocity. Therefore, in [12] the method of finite differences is used to integrate the exact Navier–Stokes partial differential equations describing the laminar flow in the inlet section of a channel with parallel walls. The distribution of axial velocities over the length and height of the channel thus obtained differs significantly from the distribution obtained by Schlichting [7] only close to the inlet at $\xi/\mathrm{Re} \le 0.0005$, since in [12] the complete system of equations of motion was used without any assumptions, and not the system of boundary layer equations. However, in view of the complexity of the calculations numerical results for the velocity distribution are presented only for $\mathrm{Re} = 300$. At $\xi/\mathrm{Re} \ge 0.0005$, i.e., outside the range of action of the leading edges of the channel walls, the velocity profiles obtained by Schlichting and the authors of [12] almost coincide.

In [13] the laminar flow in the initial section of a circular tube is calculated by numerical methods based on the boundary layer equations. A comparison of the data thus obtained with the data of [5] indicates satisfactory agreement.

The present author has attempted to refine the method of calculation proposed in [8, 9], which has not received general acceptance due to the serious errors in determining the coefficient of friction, especially on the initial section of the channel. However, if use is made of the constancy of the ratio of the conductive terms in the equation of motion $[v(\partial u/\partial y)/u(\partial u/\partial x)] \cong \mathrm{const}$, obtained in [14], this method becomes sufficiently accurate.

The boundary layer equations for stationary motion of a viscous incompressible liquid have the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

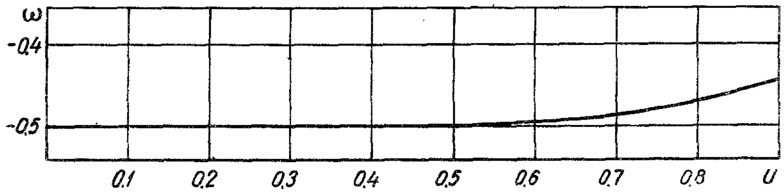

Fig. 1. Variation of the quantity ω over the thickness of the boundary layer for flow over a plate.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \quad (3)$$

We will compute the ratio of the convective terms in the equation of motion for the case of constant velocity at the outer edge of the boundary layer $U(x) = cx^m$, using the Blasius solution [15]. Figure 1 shows the quantity

$$\omega = v\frac{\partial u}{\partial y}\Big/ u\frac{\partial u}{\partial x}$$

as a function of the ratio u/U_0 in the boundary layer.

It follows from an examination of Fig. 1 that in the case of flow over a flat plate at constant velocity the quantity ω may be approximately regarded as constant over the entire thickness of the boundary layer ($u/U = 0-0.9$) and equal to some constant $\omega \cong$ const (in the given case -0.5). Then Eq. (1) takes the much simpler form

$$\varepsilon' u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}, \quad (4)$$

where

$$\varepsilon' = 1+\omega.$$

We linearize Eq. (4), replacing the longitudinal velocity component u by its mean value over the section in question U, and take this substitution into account in the newly introduced correction ε. We then obtain

$$\varepsilon U\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}. \quad (5)$$

We use this equation to solve the problem of stationary uniform (U = const) external flow over a flat plate. Then, by comparing our approximate solution with the exact Blasius solution we can calculate the correction ε.

For flow over a flat plate Eq. (4) and the boundary conditions take the form

$$\varepsilon U\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},$$

$$y=0,\quad u=0,\qquad y=\infty,\ u=U.$$

Going over to dimensionless quantities

$$\bar{u}=u/U,\ \xi=x/h,\ \eta=y/h,\ \mathrm{Re}=Uh/\nu,$$

where h is an arbitrary linear dimension, we obtain

$$\frac{\partial^2\bar{u}}{\partial\eta^2}-\varepsilon\,\mathrm{Re}\frac{\partial\bar{u}}{\partial\xi}=0,$$

$$\xi=0,\ \ \bar{u}=1,\qquad \eta=0,\ \ \bar{u}=0.$$

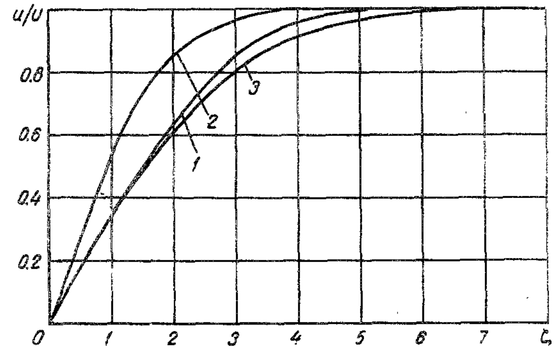

Fig. 2. Velocity profiles in the boundary layer on a flat plate: 1) Blasius solution, 2) calculation based on the method of [8, 9], 3) according to the author's formula.

The solution of this equation with the given boundary conditions is

$$\frac{u}{U}=1-\operatorname{erfc}\left(\frac{\eta}{2}\sqrt{\frac{\varepsilon\,\mathrm{Re}}{\xi}}\right).$$

Substituting the values $\eta = y/h$, $\xi = x/h$, $\mathrm{Re} = Uh/\nu$, we obtain

$$\frac{u}{U}=1-\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\varepsilon U}{\nu x}}\right).$$

Introducing the Blasius variable $\zeta = y/(\nu x/U)^{1/2}$, we find

$$\bar{u}=1-\operatorname{erfc}\left(\frac{\zeta}{2}\sqrt{\varepsilon}\right).$$

The correction ε can be found from the condition of equality of the values for the friction at the wall obtained from the Blasius solution and the proposed solution

$$\frac{\partial u}{\partial\zeta}=\sqrt{\frac{\varepsilon}{\pi}}\exp\left(-\frac{\varepsilon}{4}\zeta^2\right)\Big|_{\zeta=0}=\sqrt{\frac{\varepsilon}{\pi}}=0.33206.$$

Hence

$$\varepsilon=0.34640. \quad (6)$$

Thus, the velocity profile based on the approximate solution can finally be written in the form

$$\bar{u}=\operatorname{erf}(0.29428\,\zeta). \quad (7)$$

Calculations using this formula and comparison with the results of calculations based on the Blasius formula showed that the greatest difference occurs in the range $\zeta = 2-4$, the exact value being underestimated by 7% (Fig. 2). For comparison, the same graph includes a curve calculated on the basis of the approximate boundary layer equations presented in [9]. These

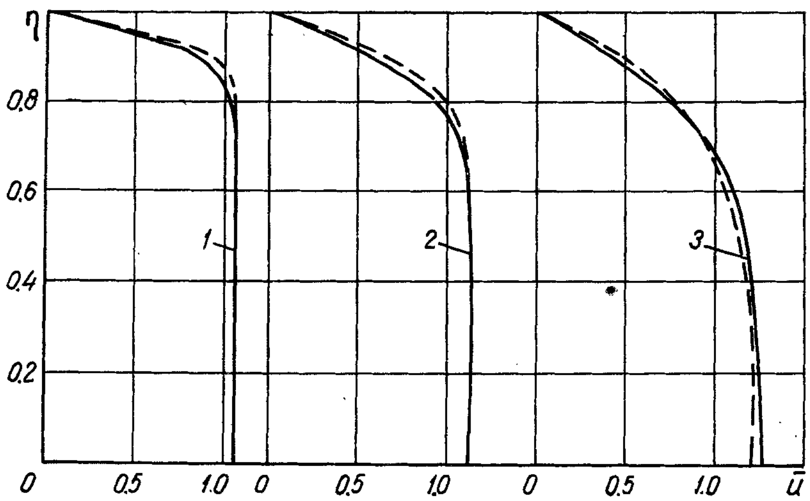

Fig. 3. Velocity distribution on initial section of a flat channel: continuous line—according to Schlichting's data [7]; broken line—according to the author's data; 1) for ξ/Re = 0.001; 2) 0.004; 3) 0.01.

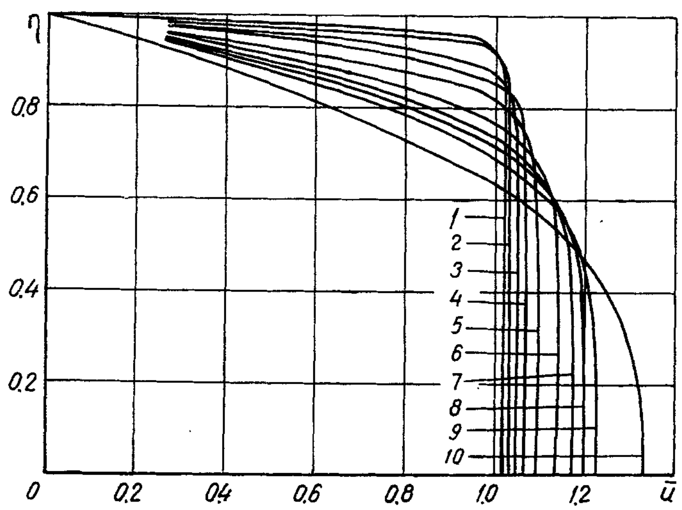

Fig. 4. Flow in the initial section of a flat channel at ξ/Re = 0.0001 (1); 0.0002 (2), 0.0005 (3); 0.001 (4); 0.002 (5); 0.004 (6); 0.006 (7); 0.008 (8) 0.01 (9); 0.02 (10).

equations differ from Eq. (5) in that they lack the correction ε and are, in the words of the author [9], a rough approximation, known to give exaggerated values of the viscosity stress.

To determine the flow in the initial section of the channel we used the following approximate equations:

$$\varepsilon U \frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{dx} + \nu \frac{\partial^2 u}{\partial y^2}, \qquad (8)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (9)$$

We will formulate the initial and boundary conditions. Let

$$u = v = 0 \quad \text{at} \quad y = \pm h,$$
$$u = U \quad \text{at} \quad x = 0. \qquad (10)$$

The rate of flow of liquid through each section of the flat channel must remain constant:

$$\int_{-h}^{h} u dy = 2hU.$$

We integrate Eq. (8) with respect to y from $-h$ to h and as a result find the value of the pressure gradient

$$\frac{1}{\rho}\frac{dp}{dx} = \frac{\nu}{2h}\left[\left(\frac{\partial u}{\partial y}\right)_{y=h} - \left(\frac{\partial u}{\partial y}\right)_{y=-h}\right]. \qquad (11)$$

Substituting the value of the pressure gradient in Eq. (8) and going over to the dimensionless quantities $\bar{u}$, ξ, η, Re, where h is half the distance between the channel walls, we obtain

$$\frac{\partial^2 \bar{u}}{\partial \eta^2} - \varepsilon \operatorname{Re}\frac{\partial \bar{u}}{\partial \xi} = \frac{1}{2}\left[\left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=1} - \left(\frac{\partial u}{\partial \eta}\right)_{\eta=-1}\right].$$

Applying a Laplace transformation to this equation we obtain

$$\frac{d^2\bar{u}^*}{d\eta^2} - \varepsilon \operatorname{Re}(s\bar{u}^* - 1) = \frac{1}{2}\left[\left(\frac{d\bar{u}^*}{d\eta}\right)_{\eta=1} - \left(\frac{d\bar{u}^*}{d\eta}\right)_{\eta=-1}\right],$$
$$\eta = \pm 1, \quad \bar{u}^* = 0. \qquad (12)$$

The solution of Eq. (12) is

$$\bar{u}^* = \frac{1}{s} - \frac{\sqrt{Hs}\operatorname{ch}(\sqrt{Hs}\,\eta) - \operatorname{sh}\sqrt{Hs}}{s(\sqrt{Hs}\operatorname{ch}\sqrt{Hs} - \operatorname{sh}\sqrt{Hs})}, \qquad (13)$$

where $H = \varepsilon \operatorname{Re}$.

For the inverse transform we obtain

$$\bar{u} = \frac{3}{2}(1-\eta^2) - $$
$$-2\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}\left(1 - \frac{\cos\mu_n\eta}{\cos\mu_n}\right)\exp\left(-\mu_n^2\frac{\xi}{H}\right), \qquad (14)$$

where μ_n are the successive roots of the equation $\operatorname{tg}\mu = \mu$.

At an infinitely large distance from the inlet of the flat channel ($\xi \to \infty$) the distribution of axial velocity over the section will be parabolic:

$$\bar{u} = \frac{3}{2}(1-\eta^2). \qquad (15)$$

Expression (14) is not convenient for small distances from the inlet since it is necessary to take too large a number of terms of the series. Accordingly, we write the expression for the transform in the following form:

$$\bar{u}^* = \frac{1}{s} - \Big\{\sqrt{Hs}\,\{\exp[-\sqrt{Hs}(1-\eta)] +$$
$$+ \exp[-\sqrt{Hs}(1+\eta)]\} -$$
$$-[1 - \exp(-2\sqrt{Hs})]\Big\}\Big\{s(\sqrt{Hs}-1)\times$$
$$\times\left[1 + \frac{\sqrt{Hs}+1}{\sqrt{Hs}-1}\exp(-2\sqrt{Hs})\right]\Big\}^{-1}.$$

Expanding the expression

$$\left[1 + \frac{\sqrt{Hs}+1}{\sqrt{Hs}-1}\exp(-2\sqrt{Hs})\right]$$

in a geometric series and confining ourselves to the first term we obtain

$$\bar{u}^* = \frac{1}{s} + \{\sqrt{Hs}\,[\exp(-\sqrt{Hs}(1-\eta)) +$$
$$+ \exp(-\sqrt{Hs}(1+\eta))] - [1 - \exp(-2\sqrt{Hs})]\}\times$$
$$\times\{s(1-\sqrt{Hs})\}^{-1}.$$

Passing to the inverse transform we obtain an expression for the velocity distribution that is valid at small distances from the inlet:

$$\bar{u} = \exp(\xi/H)\left(1 + \operatorname{erf}\sqrt{\frac{\xi}{H}}\right) - \exp\left[\frac{\xi}{H} - (1-\eta)\right]\times$$
$$\times \operatorname{erfc}\left(-\sqrt{\frac{\xi}{H}} + \frac{1-\eta}{2}\sqrt{\frac{H}{\xi}}\right) - \exp[\xi/H - (1+\eta)]\times$$
$$\times \operatorname{erfc}\left(-\sqrt{\frac{\xi}{H}} + \frac{1+\eta}{2}\sqrt{\frac{H}{\xi}}\right) + \operatorname{erfc}\sqrt{\frac{H}{\xi}}. \qquad (16)$$

On the initial section of the channel at $\xi/\operatorname{Re} \leq 0.01$ and at $\eta \geq 0$, i.e., in the upper half of the channel, the velocity profile can be calculated from the following simplified formula:

$$\bar{u} = \exp(\xi/H)\left(1 + \operatorname{erf}\sqrt{\frac{\xi}{H}}\right) - \exp\left[\frac{\xi}{H} -\right.$$
$$\left.-(1-\eta)\right]\operatorname{erfc}\left(-\sqrt{\frac{\xi}{H}} + \frac{1-\eta}{2}\sqrt{\frac{H}{\xi}}\right). \qquad (17)$$

It is clear from Fig. 3 that the data obtained in [7] and in the present study differ by not more than 10%.

Thus, the proposed method of calculating the velocity fields in the initial section of a flat channel is in quite good agreement with previously developed methods and, moreover, gives a simple analytic expression for the velocity profile and the coefficient of friction. By way of example, Fig. 4 shows the velocity profiles at different sections of a flat channel.

On the basis of the analytic expression for the velocity distribution (14) it is possible to find the dimensionless shear stress at the wall

$$\frac{\tau_0}{\rho U^2} = \frac{1}{\mathrm{Re}} \frac{\partial \bar{u}}{\partial \eta}\bigg|_{\eta=1} = \left[3+2\sum_{n=1}^{\infty} \exp\left(-\mu_n^2 \frac{\xi}{\varepsilon \mathrm{Re}}\right)\right] \Big/ \mathrm{Re}. \quad (18)$$

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